Applications of the Light Cone Condition for various perturbed Vacua.[*]

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Abstract We examine the propagation of light in the presence of various modifications of the QED vacuum in the limit of low frequency. A polarization summed and direction averaged-light cone condition is derived from the equation of motion which arises from the effective QED action. Several applications are given concerning vacuum modifications caused by, e.g., strong fields, Casimir systems and finite temperature.

Table of Contents

1 Modification of the Vacuum Light Velocity $c(=1)$ in Presence of External Fields, Finite Temperature, Casimir Plates, etc. Examples.

In this first chapter we want to list some examples which demonstrate how the light velocity $c(=1)$ becomes changed when a light ray traverses a region which is disturbed by the presence of certain field configurations, finite temperature environment, or Casimir plates.

(a) Weak electromagnetic fields.

Let us focus on a pure constant applied magnetic field $\mathbf{B}$ which acts as a birefringent medium for the incoming photon beam. One can distinguish between two polarization modes where either the $\mathbf{e}$-field ($\perp$-mode) or the $\mathbf{h}$-field ($\|$-mode) of the light wave points along the direction perpendicular to the plane containing the $\mathbf{B}$-field and the wave propagation direction , $(\hat{\mathbf{B}}, \hat{\mathbf{k}})$-plane. Then we obtain two corresponding refractive indices

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$\left(\frac{k}{\omega}\right)_{\perp,\parallel}$ which are given by [1]:

$$\begin{aligned} n_\perp &= 1+\frac{8\alpha^2}{45m^4}\left(1+\frac{40}{9}\frac{\alpha}{\pi}\right)B^2\,\sin^2\theta, \qquad \theta=\sphericalangle(\mathbf{B},\mathbf{k}) \\ n_\parallel &= 1+\frac{14\alpha^2}{45m^4}\left(1+\frac{1315}{252}\frac{\alpha}{\pi}\right)B^2\,\sin^2\theta. \end{aligned} \tag{1.1}$$

The change in the light velocity is correspondingly $\left(v_{\parallel,\perp}=\frac{c(=1)}{n_{\parallel,\perp}}\right)$

$$\begin{aligned} v_\perp &= 1-\frac{8\alpha^2}{45m^4}\left(1+\frac{40}{9}\frac{\alpha}{\pi}\right)B^2\,\sin^2\theta, \qquad \theta=\sphericalangle(\mathbf{B},\mathbf{k}) \\ v_\parallel &= 1-\frac{14\alpha^2}{45m^4}\left(1+\frac{1315}{252}\frac{\alpha}{\pi}\right)B^2\,\sin^2\theta. \end{aligned} \tag{1.2}$$

So, when we consider light of wavelength λ travelling a path of length L normal to the **B**-field, the angular rotation of the plane of polarization is given by [2]

$$\Psi_{\mathrm{QED}}=\frac{1}{15}\alpha\left(\frac{eB}{m^2}\right)^2\frac{L}{\lambda}\left(1+\frac{25}{4}\frac{\alpha}{\pi}\right). \tag{1.3}$$

Using the two phase velocities (1.2) (to one-loop order only) the average over polarization and direction is given by

$$v = \frac{1}{4\pi}\int d\Omega\,\frac{1}{2}\left(v_\perp+b_\parallel\right)=1-\frac{22}{135}\frac{\alpha^2}{m^4}B^2 \tag{1.4}$$

$$\text{or}\quad \delta v = -\frac{44}{135}\frac{\alpha^2}{m^4}u, \qquad u=\frac{1}{2}B^2. \tag{1.5}$$

(b) Temperature-induced velocity shift.

Here the velocity of soft photons moving in a photon gas at low temperature $T \ll m$ is given by [3]

$$v=1-\frac{44\pi^2}{2025}\alpha^2\frac{T^4}{m^4}. \tag{1.6}$$

(c) Casimir vacuum.

For a photon propagating perpendicular to Casimir plates with distance a one obtains [4]

$$v=1+\frac{11}{(90)^2}\frac{\alpha^2}{m^4}\frac{\pi^2}{a^4}>1 \qquad ! \tag{1.7}$$

Further velocity shifts can be found for photons moving in a gravitational background [5]. Observe that all the examples are low-energy phenomena.

Covering all these aforementioned cases Latorre, Pascual and Tarrach [6] presented an intriguing general, so-called "unified" formula. They claimed that the polarization and direction-averaged velocity shift is related to the (renormalized) background energy density u with a "universal" numerical coefficient

$$\delta v = -\frac{44}{135}\frac{\alpha^2}{m^4}\, u. \tag{1.8}$$

That all these cases can only be of limited validity becomes clear when looking at the strong $\mathbf{B}$-field regime ($B > B_{\rm cr} = \frac{m^2}{e}$) where one finds [7]

$$v^2 = 1 - \frac{\alpha}{4\pi}\sin^2\theta \left[\frac{2}{3}\frac{B}{B_{\rm cr}} + \mathcal{O}(1) + \mathcal{O}\left(\frac{B_{\rm cr}}{B}\ln\frac{B}{B_{\rm cr}}\right)\right]. \tag{1.9}$$

Also of limited value is Shore's conjecture [8] that the "universal" coefficient in (1.8) can be related to the trace anomaly of the energy momentum tensor:

$$\langle T^\alpha{}_\alpha\rangle_{\rm E.M.} = -4\left[\underbrace{\frac{8}{45}\frac{\alpha^2}{m^4}}_{\sim\delta v_\perp}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)^2 + \underbrace{\frac{14}{45}\frac{\alpha^2}{m^4}}_{\sim\delta v_\parallel}\left(\frac{1}{4}{}^\star F_{\mu\nu}F^{\mu\nu}\right)^2\right]. \tag{1.10}$$

It is our goal to rederive some of the former results and generalize the "unified formula" of Latorre, Pascual and Tarrach. We confine ourselves to the case of non-trivial vacua modified by QED phenomena.

2 Light Cone Condition (LCC): Effective Action Approach [9].

Here are the essential assumptions that are needed for a description of light propagation in various perturbed vacua employing the effective action approach:
(1) The propagating photons with field strength $f^{\mu\nu}$ are considered to be soft: $\frac{\omega}{m} \ll 1$.
(2) The vacuum modification is homogeneous in space and time.
(3) Vacuum modifications caused by the propagating light itself are negligible.

Let us now turn to pure electromagnetic vacuum modifications where the dynamical building blocks of the effective action which respect Lorentz and gauge invariance are given by

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, & (2.1)\\ {}^\star F^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. & (2.2)\end{aligned}$$

The lowest-order linearly independent scalars (pseudo-scalars) are

$$\begin{aligned} x &:= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\big(\mathbf{B^2} - \mathbf{E^2}\big) \qquad \big(=\mathcal{F}\big), & (2.3)\\ y &:= \frac{1}{4}{}^\star F_{\mu\nu}F^{\mu\nu} = -\mathbf{E}\cdot\mathbf{B} \qquad\qquad \big(=\mathcal{G}\big). & (2.4)\end{aligned}$$

Hence the Maxwell Lagrangian can be written as $\mathcal{L}_{\mathrm{M}} = -x$. The fundamental algebraic relations [10],

$$\begin{aligned} F^{\mu\alpha} F^{\nu}{}_{\alpha} - {}^\star F^{\mu\alpha}\, {}^\star F^{\nu}{}_{\alpha} &= 2\, x\, g^{\mu\nu}\,, \\ F^{\mu\alpha}\, {}^\star F^{\nu}{}_{\alpha} = {}^\star F^{\mu\alpha} F^{\nu}{}_{\alpha} &= y\, g^{\mu\nu}, \end{aligned} \tag{2.5}$$

help to verify (i) the vanishing of odd-order invariants and (ii) that invariants of arbitrary order can be reduced to expressions only involving $x^n y^m$; $n, m = 0, 1, 2 \ldots$. Besides, note that parity invariance demands for m to be even. The corresponding Lagrangian becomes then simply a function of x and y:

$$\mathcal{L} = \mathcal{L}(x, y). \tag{2.6}$$

From here we obtain the equations of motion by variation,

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\mu \big(\partial_x \mathcal{L}\, F^{\mu\nu} + \partial_y \mathcal{L}\, {}^\star F^{\mu\nu} \big), \tag{2.7}$$

or, after taking advantage of the Bianchi identity while moving ∂_μ to the right,

$$0 = (\partial_x \mathcal{L})\, \partial_\mu F^{\mu\nu} + \left(\frac{1}{2} M^{\mu\nu}_{\alpha\beta} \right)\, \partial_\mu F^{\alpha\beta}, \tag{2.8}$$

where

$$M^{\mu\nu}_{\alpha\beta} := F^{\mu\nu} F_{\alpha\beta}\, (\partial_x^2 \mathcal{L}) + {}^\star F^{\mu\nu}\, {}^\star F_{\alpha\beta}\, (\partial_y^2 \mathcal{L}) + \partial_{xy} \mathcal{L} \left(F^{\mu\nu}\, {}^\star F_{\alpha\beta} + {}^\star F^{\mu\nu} F_{\alpha\beta} \right). \tag{2.9}$$

What follows is a 5-step calculation consisting of

(1) $\quad F^{\mu\nu} \longrightarrow \underbrace{F^{\mu\nu}}_{\text{const. background field}} + \underbrace{f^{\mu\nu}}_{\text{propag. light wave}} \quad : \quad \partial_\mu F^{\kappa\lambda} \longrightarrow \partial_\mu f^{\kappa\lambda}$

(2) $\quad f^{\mu\nu} = k^\mu a^\nu - k^\nu a^\mu = a(k^\mu \epsilon^\nu - k^\nu \epsilon^\mu)$ where $k_\mu \epsilon^\mu = 0$ in the Lorentz gauge

(3) $\quad$ Average over polarization states: $\sum_{\text{pol.}} \epsilon^\beta \epsilon^\nu \rightarrow g^{\beta\nu}$.

These intermediate steps yield instead of (2.8)

$$0 = 2(\partial_x \mathcal{L})\, k^2 + M^{\mu\nu}_{\alpha\nu}\, k_\mu k^\alpha. \tag{2.10}$$

Equation (2.10) already represents a light cone condition and indicates that the familiar $k^2 = 0$ relation will in general not hold for arbitrary Lagrangians.

(4) $\quad$ Using the fundamental relations (2.5) together with the Maxwell energy-momentum tensor,

$$T^{\mu}{}_{\alpha} = F^{\mu\nu} F_{\alpha\nu} - x\, \delta^\mu_\alpha, \tag{2.11}$$

we obtain

$$M^{\mu\nu}_{\alpha\nu} = 2\left[\frac{1}{2}T^{\mu}{}_{\alpha}(\partial_x^2+\partial_y^2)\mathcal{L} + \delta^{\mu}_{\alpha}\left(\frac{1}{2}x(\partial_x^2-\partial_y^2)\mathcal{L} + y\partial_{xy}\mathcal{L}\right)\right]. \tag{2.12}$$

(5) Finally we need the VEV of the energy-momentum tensor

$$\langle T^{\mu\nu}\rangle_{xy} = -T^{\mu\nu}(\partial_x\mathcal{L}) + g^{\mu\nu}\left(\mathcal{L} - x\partial_x\mathcal{L} - y\partial_y\mathcal{L}\right) =: \frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}}, \tag{2.13}$$

where $\Gamma := \int d^4x\,\sqrt{-g}\,\mathcal{L}$ denotes the effective action.

After these 5 steps we end up with the desired LCC:

$$k^2 \;=\; Q\,\langle T^{\mu\nu}\rangle_{xy}\,k_\mu k_\nu, \tag{2.14}$$

with

$$Q = \frac{\frac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}}{\left[(\partial_x\mathcal{L})^2 + (\partial_x\mathcal{L})\left(\frac{x}{2}(\partial_x^2-\partial_y^2)+y\partial_{xy}\right)\mathcal{L} + \frac{1}{2}(\partial_x^2+\partial_y^2)\mathcal{L}(1-x\partial_x-y\partial_y)\mathcal{L}\right]}. \tag{2.15}$$

Now we want to extend the LCC to arbitrary vacuum disturbances, not only of electromagnetic field type. Let us therefore parametrize the additional vacuum modifications by the label z and write instead of (2.14)

$$\begin{aligned} k^2 &= {}_z\langle 0|\,Q\,\langle T^{\mu\nu}\rangle_{xy}\,|0\rangle_z\,k_\mu k_\nu \\ &= \sum_i {}_z\langle 0|\,Q\,|i\rangle_z\,{}_z\langle i|\langle T^{\mu\nu}\rangle_{xy}\,|0\rangle_z\,k_\mu k_\nu. \end{aligned} \tag{2.16}$$

In the sequel, we assume the vacuum to behave as a passive medium which leads to

$${}_z\langle 0|\,Q\,|i\rangle_z = \langle Q\rangle_z\,\delta_{0i}\,. \tag{2.17}$$

This equation states that the vacuum exhibits no back-reaction caused by the EM fields while switching on z. One can think of this approximation as a kind of adiabatic or Born-Oppenheimer approximation.

Q depends functionally on $\mathcal{L}(x,y) \equiv \mathcal{L}^{\mathrm{eff}}(A^{\mathrm{ext}})$, which is, as usual, defined via the functional integral over the fluctuating fields:

$$\mathrm{e}^{\mathrm{i}\int d^4q\,\mathcal{L}^{\mathrm{eff}}(A^{\mathrm{ext}})} = \int\left[d\psi d\bar\psi dA\right]\,\exp\Big\{\mathrm{i}S_{\mathrm{QED}}\left[\psi,\bar\psi,A;A^{\mathrm{ext}}\right]\Big\} = Z\left[A^{\mathrm{ext}}\right].$$

From here we obtain for the definition of $\langle\mathcal{L}^{\mathrm{eff}}\rangle_z \equiv \mathcal{L}(x,y;z)$

$$\mathrm{e}^{\mathrm{i}\int d^4q\,\mathcal{L}(x,y;z)} = \int\limits_z\left[d\psi d\bar\psi dA\right]\,\exp\Big\{\mathrm{i}S_{\mathrm{QED}}\left[\psi,\bar\psi,A;A^{\mathrm{ext}}\right]\Big|_z\Big\} = Z\left[A^{\mathrm{ext}}\right]\Big|_z.$$

E.g., if the modification z imposes boundary conditions on the fields, as is the case for the Casimir effect or the thermalization of photons or fermions in loop graphs, the functional integral has to be taken over the fields which satisfy these boundary conditions. Therefore, taking the VEV of Q defines the new effective Lagrangian characterizing the complete modified vacuum:

$$\langle Q \rangle_z = \langle Q(\mathcal{L}(x,y)) \rangle_z = Q(\mathcal{L}(x,y;z))\,. \tag{2.18}$$

Here, then, is the LCC for arbitrary homogeneous modified vacua:

$$k^2 = Q(x,y,z)\, \langle T^{\mu\nu} \rangle_{xyz}\, k_\mu k_\nu\,. \tag{2.19}$$

If we choose a special Lorentz frame,

$$\bar{k}^\mu = \frac{k^\mu}{|\mathbf{k}|} = \left(\frac{k^0}{|\mathbf{k}|}, \hat{\mathbf{k}} \right) =: (v, \hat{\mathbf{k}}), \tag{2.20}$$

we obtain for Eq. (2.19):

$$v^2 = 1 - Q\, \langle T^{\mu\nu} \rangle \bar{k}_\mu \bar{k}_\nu. \tag{2.21}$$

This LCC is a generalization of the "unified formula" of Latorre, Pascual and Tarrach [6].

If we, furthermore, average over propagation directions, i.e., integrate over $\hat{\mathbf{k}} \in S^2$ and assume $Q\langle T^{00}\rangle, \langle T^\alpha{}_\alpha \rangle \ll 1$, we obtain:

$$v^2 = 1 - \frac{4}{3} Q\, \langle T^{00} \rangle = 1 - \frac{4}{3} Q\, u. \tag{2.22}$$

At this stage let us return to Shore's conjecture [8] which suggests a deeper connection between the velocity shift and the trace anomaly. From (2.13), we can read off the relation

$$\langle T^\alpha{}_\alpha \rangle = 4(\mathcal{L} - x\partial_x \mathcal{L} - y\partial_y \mathcal{L}) \tag{2.23}$$

$$\overset{\text{H.E.}}{=} -4 \left[\underbrace{\frac{8}{45}\frac{\alpha^2}{m^4}}_{\sim \delta v_\perp} x^2 + \underbrace{\frac{14}{45}\frac{\alpha^2}{m^4}}_{\sim \delta v_\parallel} y^2 \right]. \tag{2.24}$$

So there is a relation between the trace anomaly and the velocity shift for different polarization states. Notice, however, that this result is tied to using the Heisenberg-Euler Lagrangian in the weak field limit. In general it is the Q-factor that is linked to $\langle T^\alpha{}_\alpha \rangle$, as can be seen by writing the numerator of (2.15) in the form

$$\mathrm{num}(Q) = \frac{1}{2}(\partial_x^2 + \partial_y^2)\mathcal{L} = -\frac{1}{2}\left(\frac{y}{x} + \frac{x}{y} \right) \partial_{xy}\mathcal{L} - \frac{1}{8}\left(\frac{1}{x}\partial_x + \frac{1}{y}\partial_y \right) \langle T^\alpha{}_\alpha \rangle \tag{2.25}$$

and only for the special structure of the H.E.-Lagrangian where $\partial_{xy}\mathcal{L} = 0$ is Shore's conjecture true. Referring to the right-hand side of (2.21), the value and sign of the velocity shift result from the competition between the VEV of the energy-momentum tensor and the Q-factor. For small corrections to the Maxwell Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm M} + \mathcal{L}_{\rm c}, \tag{2.26}$$

we have denom$(Q) = 1 + \mathcal{O}(\mathcal{L}_{\rm c})$, which reduces Eq. (2.15) to

$$Q \simeq \frac{1}{2}(\partial_x^2 + \partial_y^2)\mathcal{L} \quad \Longrightarrow \nabla^2\mathcal{L} = 2\,Q\,. \tag{2.27}$$

Because of the similarity to the (2D) Poisson equation, from now on we will call Q the effective action charge in field space. The classical vacuum $\mathcal{L}_{\rm M} = -x$ is uncharged and hence $v = 1$. As we will soon demonstrate, the pure QED vacuum has a small positive charge at the origin in field space ($x = 0 = y$).

3 Applications of the LCC.

A. Weak EM fields.

Let us start with the two-loop Heisenberg-Euler Lagrangian [11],

$$\mathcal{L} = -x + c_1\,x^2 + c_2\,y^2, \tag{3.1}$$

with

$$\begin{aligned} c_1 &= \frac{8\alpha^2}{45m^4}\left(1 + \frac{40}{9}\frac{\alpha}{\pi}\right), \\ c_2 &= \frac{14\alpha^2}{45m^4}\left(1 + \frac{1315}{252}\frac{\alpha}{\pi}\right). \end{aligned} \tag{3.2}$$

Then we obtain for the effective action charge

$$Q = \frac{1}{2}(\partial_x^2 + \partial_y^2)\mathcal{L} = c_1 + c_2, \tag{3.3}$$

so that Eq. (2.22) immediately yields:

$$v = 1 - \frac{4\alpha^2}{135m^4}\left(11 + \frac{1955}{36}\frac{\alpha}{\pi}\right)\left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right]. \tag{3.4}$$

3.1 B. Strong fields

Here we begin with Schwinger's proper-time expression [10] – still valid for arbitrary constant field strength –

$$\mathcal{L} = -x - \frac{1}{8\pi^2}\int_0^{i\infty}\frac{ds}{s^3}e^{-m^2 s}\Big[(es)^2|y|\coth\Big(es\big(\sqrt{x^2+y^2}+x\big)^{\frac{1}{2}}\Big)\cot\Big(es\big(\sqrt{x^2+y^2}-x\big)^{\frac{1}{2}}\Big) - \frac{2}{3}(es)^2 x - 1\Big]. \tag{3.5}$$

The complete formula for the effective action charge for a purely magnetic background field can be performed analytically and yields $\big(h = \frac{B_{\rm cr}}{2B}\big)$:

$$Q(h) = \frac{1}{2B^2}\frac{\alpha}{\pi}\Big[\Big(2h^2 - \frac{1}{3}\Big)\Psi(1+h) - h - 3h^2 - 4h\ln\Gamma(h) + 2h\ln 2\pi + \frac{1}{3} + 4\zeta'(-1,h) + \frac{1}{6h}\Big]. \tag{3.6}$$

For strong fields, the last term in (3.6), $\propto \frac{1}{6h} \propto B$, dominates the expression in the square brackets. Hence, the effective action charge decreases with

$$Q(B) \simeq \frac{1}{6}\frac{\alpha}{\pi}\frac{1}{B_{\rm cr}}\frac{1}{B}, \qquad \text{for} \quad B \to \infty. \tag{3.7}$$

Finally, the light cone condition (2.21) yields, for strong magnetic background fields for which $B \gtrsim B_{\rm cr}$.

$$\begin{aligned} v^2 &= 1 - \frac{\alpha}{\pi}\frac{\sin^2\theta}{2}\ \Big[\Big(\tfrac{B_{\rm cr}^2}{2B^2} - \tfrac{1}{3}\Big)\psi\big(1+\tfrac{B_{\rm cr}}{2B}\big) - \tfrac{2B_{\rm cr}}{B}\ln\Gamma\big(\tfrac{B_{\rm cr}}{2B}\big) - \tfrac{3B_{\rm cr}^2}{4B^2} \\ &\qquad - \tfrac{B_{\rm cr}}{2B} + \tfrac{B_{\rm cr}}{B}\ln 2\pi + \frac{1}{3} + 4\zeta'\big(-1,\tfrac{B_{\rm cr}}{2B}\big) + \tfrac{B}{3B_{\rm cr}}\Big], \end{aligned} \tag{3.8}$$

$$\longrightarrow 1 - \frac{\alpha}{4\pi}\sin^2\theta\frac{2}{3}\frac{B}{B_{\rm cr}} \qquad \text{for} \quad B \gtrsim B_{\rm cr}, \tag{3.9}$$

$$\text{e.g.,} \quad \delta v \simeq 9.58..\cdot 10^{-5} \qquad \text{at} \quad B = B_{\rm cr} = \frac{m^2}{e}, \tag{3.10}$$

At least on a formal stage, even the $B \to \infty$ limit can be taken. For this, the complete structure of the effective action charge Q in Eq. (2.15) has to be taken into account as well as the phase velocity dependence of the product $\langle T^{\mu\nu}\rangle \bar{k}_\mu \bar{k}_\nu$ in Eq. (2.22). The result for the velocity square reads:

$$v^2 = 1 - \sin^2\theta + \frac{6\pi}{\alpha}\frac{B_{\rm cr}}{B}\sin^2\theta + \ldots, \qquad \text{for} \quad \frac{B}{B_{\rm cr}} \to \infty. \tag{3.11}$$

In this limit, we find that a photon propagation perpendicular to the magnetic field is strongly suppressed, while a propagation along the magnetic field lines obtains no

modifications. However, we must remember that we calculated the polarization-summed average velocity, and so the true modes of propagation might be different. Nevertheless, this result shows similarities to the propagation of photons along the magnetic field lines in a plasma (Alfvén-mode propagation).

C. Casimir Vacua (Scharnhorst effect [4]).

In accordance with experimental facilities, the plate separation a is treated as a macroscopic parameter ($a \propto \mu$m). In order not to violate the soft photon approximation $m \gg \frac{1}{\lambda}$, the photon wavelength has to obey $\lambda \ll a$. Only then can we treat the Casimir region as a (macroscopic) medium.

Accepting these assumptions, Q is simply given by

$$\begin{aligned} Q &= Q(x,y,a)\Big|_{x=0=y} = Q(x,y,0)\Big|_{x=0=y} + \underbrace{\Delta Q(x,y,a)}_{\sim \mathrm{e}^{-ma}}\Big|_{x=0=y} \overset{ma\gg 1}{\simeq} Q(x,y)\Big|_{x=0=y} \\ &= c_1 + c_2 = \frac{2}{45}\frac{\alpha^2}{m^4}\left(11 + \frac{1955}{36}\frac{\alpha}{\pi}\right). \end{aligned} \tag{3.12}$$

Together with $\langle T^{\mu\nu}\rangle$, as given in Ref. [12],

$$\langle T^{\mu\nu}\rangle = \frac{\pi^2}{720a^4}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} \tag{3.13}$$

and Eq. (2.21), we obtain for the propagation of light perpendicular to the plates the superluminal phase and group velocity:

$$v = 1 + \frac{1}{(90)^2}\frac{\alpha^2}{m^4}\left(11 + \frac{1955}{36}\frac{\alpha}{\pi}\right)\frac{\pi^2}{a^4}. \tag{3.14}$$

Equation (3.14) represents the two-loop corrected version of Scharnhorst's formula.[1].

Taking the leading radiative correction to the Casimir energy into account, Bordag, Robaschik and Wieczorek [13] obtained the result:

$$\langle T^{00}\rangle \equiv u = -\frac{\pi^2}{720a^4} + \frac{1}{2560}\frac{\alpha}{\pi}\frac{\pi^3}{ma^5}. \tag{3.15}$$

At the two-loop level of Eq. (3.14), this correction can nevertheless be neglected, since $ma \gg 1$.

[1]Potentially, there might be a further contribution to the two-loop correction, since the Casimir boundary conditions for the radiative photon have not been taken into account in the calculation of the two-loop Lagrangian.

D. Finite Temperature.

We begin with the one-loop correction to the effective QED Lagrangian at finite temperature which can be decomposed according to

$$\mathcal{L} = \mathcal{L}_{\mathrm{M}} + \mathcal{L}(T=0) + \Delta\mathcal{L}(T), \tag{3.16}$$

from which follows:

$$Q = Q(T=0) + \Delta Q(T). \tag{3.17}$$

For purely magnetic fields, $\Delta\mathcal{L}$ was calculated by Dittrich [14]:

$$\Delta\mathcal{L}(B,T) = -\frac{\sqrt{\pi}}{4\pi^2}\int\limits_0^{\mathrm{i}\infty}\frac{ds}{s^{\frac{5}{2}}}\mathrm{e}^{-m^2 s} esB\cot esB\; T\left[\Theta_2(0,4\pi\mathrm{i}sT^2) - \frac{1}{2T\sqrt{\pi s}}\right]. \tag{3.18}$$

Notice that in $Q = \frac{1}{2}\big(\partial_x^2 + \partial_y^2\big)\mathcal{L}$, we have to differentiate with respect to x and y. So we must first re-introduce x and y, differentiate and then put $\mathbf{E} = 0$, i.e.,

$$esB\cot esB \quad\longrightarrow\quad (es)^2|y|\coth\Big[es\big(\sqrt{x^2+y^2}+x\big)^{\frac{1}{2}}\Big]\cot\Big[es\big(\sqrt{x^2+y^2}-x\big)^{\frac{1}{2}}\Big].$$

The temperature-dependent part of the effective action charge is then given by

$$\Delta Q(B=0,T) = \frac{22}{45}\frac{\alpha^2}{m^4}\sum_{n=1}^{\infty}(-1)^n\left(\frac{m}{T}n\right)^2 K_2\left(\tfrac{m}{T}n\right). \tag{3.19}$$

Here are some limiting cases. For low temperature we obtain:

$$\Delta Q(B=0,T\to 0) \simeq -\frac{22}{45}\frac{\alpha^2}{m^4}\sqrt{\frac{\pi}{2}}\left(\frac{m}{T}\right)^{\frac{3}{2}}\mathrm{e}^{-\frac{m}{T}} \quad\longrightarrow\quad 0^-. \tag{3.20}$$

Hence, the effective action charge is perfectly described by

$$Q(B=0,T=0) = c_1 + c_2 = \frac{22}{45}\frac{\alpha^2}{m^4}.$$

In the high-temperature limit $\frac{T}{m} \gg 1$, the result turns out to be

$$\Delta Q(T\gg m) = -\frac{22}{45}\frac{\alpha^2}{m^4}\left[1 - \frac{k_1}{4}\frac{m^4}{T^4} + \mathcal{O}\left(\frac{m^6}{T^6}\right)\right], \qquad k_1 = 0.123749\ldots. \tag{3.21}$$

Therefore the complete effective action charge decreases rapidly, $\propto \frac{1}{T^4}$:

$$Q(T\gg m) = Q(T=0) + \Delta Q(T\gg m) = \frac{11}{90}k_1\frac{\alpha^2}{T^4} + \mathcal{O}\left(\tfrac{m^2}{T^6}\right). \tag{3.22}$$

For the LCC we need the VEV of the energy-momentum tensor which is given by [15]

$$\langle T_{\mu\nu}\rangle_T = \frac{\pi^2}{90}\,(N_{\rm B} + \tfrac{7}{8}N_{\rm F})\,T^4\ \mathrm{diag}(3,1,1,1). \tag{3.23}$$

For QED, we obtain:

$$\begin{array}{llll} N_{\rm B} = 2 \quad , & N_{\rm F} = 0, & \text{for} \quad T \ll m & \text{(photon gas)} \\ N_{\rm B} = 2 \quad , & N_{\rm F} = 4, & \text{for} \quad T \gg m & \text{(photon + } e^+e^-\text{-fermion gas).} \end{array}$$

So we arrive at the velocity of soft photons moving in a photon (or photon + e^+e^--) gas:

$$\text{low T}: \qquad v = 1 - \frac{44\pi^2}{2025}\frac{\alpha^2}{m^4}T^4 \qquad T < 10^6\,\mathrm{K} \tag{3.24}$$

$$\text{high T}: \qquad v = 1 - \frac{121}{8100}\,k_1\pi^2\alpha^2 = 1 - 9.72\cdot 10^{-7} = \text{const.} \tag{3.25}$$

E. Casimir Vacua at Finite Temperature.

First we consider the low-temperature region. Here the LCC yields:

$$v = 1 + \frac{1}{(90)^2}\frac{\alpha^2}{m^4}\left(11 + \frac{1955}{36}\frac{\alpha}{\pi}\right)\frac{\pi^2}{a^4}\left(1 - \frac{180\zeta(3)}{\pi^4}(Ta)^3\right) \quad \text{for} \quad (Ta)\to 0. \tag{3.26}$$

Neglecting $(Ta)^3$ in the low-temperature limit we end up with Scharnhorst's result. But we do not find an additional velocity shift $\sim T^4$, as could have been expected from Eq. (3.24). This clearly arises from the fact that none of the (quantized) perpendicular modes can be excited at low temperature. The $(Ta)^3$-term in Eq. (3.26) will become important for $(Ta) = \mathcal{O}(1)$, i.e., $T > 2000$ K. This shows that the Scharnhorst effect is stable for $T < 2000$ K.

For increasing temperature, we find an intermediate temperature region characterized by the condition $1 \ll Ta \ll ma$ which corresponds to $0.2\mathrm{eV} < T < 0.5\mathrm{MeV}$. This implies that $Q = Q(T=0)$ is a justified approximation. The velocity shift in this case is given by

$$v = 1 - \frac{4\pi^2}{(45)^2}\frac{\alpha^2}{m^4}\left(11 + \frac{1955}{36}\frac{\alpha}{\pi}\right)T^4\left(1 - \frac{45\zeta(3)}{16\pi^3}\frac{1}{(Ta)^3}\right). \tag{3.27}$$

In this limit, only the modifications caused by the blackbody radiation become important. A term proportional to $\frac{1}{a^4}$ does not occur, since higher (perpendicular) modes have been excited.

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